

VARIATIONAL METHODS

In recent years, the calculus of variation has been used more conveniently in the formulation of finite element method. Such an approach makes it possible to apply finite element technique to solve field problems. Therefore, it is necessary to fully understand the basics of variational method such as calculus of variation.

1. Functional: A function of other function is called functional. More specifically a functional is defined as an expression that takes on a particular value which is dependent on the function used in the functional. The functional is represented in the form of integral with first or higher order derivatives and is given by

$$I = \int_{x_1}^{x_2} F(x, u, u', u'') dx$$

where the variables u and its first and second derivatives w.r.t x, u' and u'' are functions of x . The functions F and I are called functionals because they are functions of u, u', u'' and u, u', u'' are function of other function x .

The potential energy of a deformable body is an example for the functional because the total potential energy π is a function of deformation u and u is a function of other function x .

For example, in P.E of a 1 – D bar $\pi = \frac{1}{2} Ku^2 - Pu$, π is a function of deformation u and u is a function of x .

2. Calculus of Variations:

The calculus of variation in FEM is used to derive the governing differential equation and natural and geometric boundary conditions of the problem. In this calculus of variation, we concerned with functionals and their extremum values. The functional is expressed as

$$I = \int_{x_1}^{x_2} F(x, u, u', u'') dx \quad \text{---(a)}$$

where the variables u and its first and second derivatives w.r.t x, u' and u'' are functions of x . The functions F and I are called functionals because they are functions of other function. The integral in Eqn. (a) is defined between the two points x_1 and x_2 as shown in Fig. 1. The corresponding values of u between these points will be

$$u(x_1) = u_1$$

$$u(x_2) = u_2$$

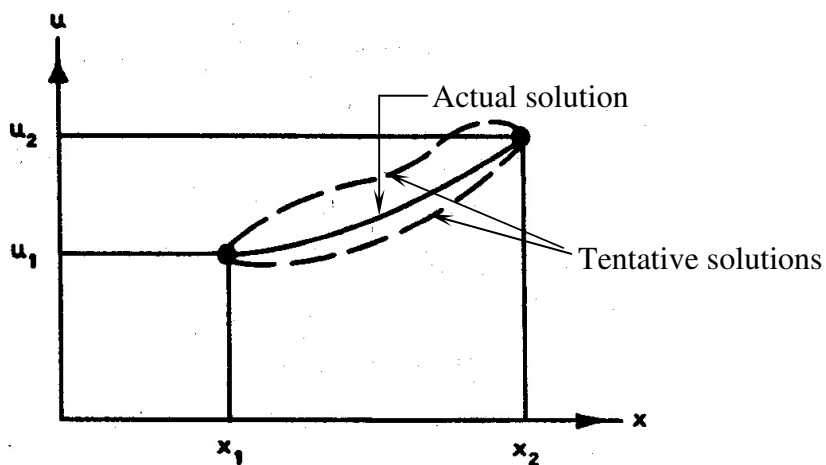


Fig. 1: Domain between x_1 and x_2

In variational procedure a tentative solution is tried for a given problem and the functional is expressed in terms of this tentative solution. From all such possible solutions satisfying the boundary conditions, the solution which satisfies the variational principle will be the one which makes the functional I stationary (i.e. maximum or minimum). The mathematical procedure used to select the correct solution from a number of tentative solutions is the *calculus of variation*.

3. Derivation of Euler-Lagrange Equation and Equations for Boundary Conditions:

Consider a functional $I = \int_{x_1}^{x_2} F(x, u, u', u'') dx$ ---(a)

Let a tentative solution to the above functional be represented as $\bar{u}$ and it is the sum of exact solution u and its variation δu as shown in Fig. 2.

i.e. $\bar{u} = u + \delta u$

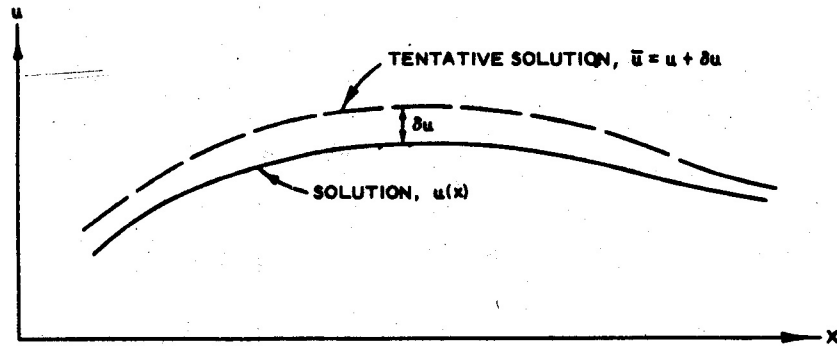


Fig. 2: Variational notation

It may be noted that, the notation δ used in variational calculus is similar to differential operator d used in differential calculus. Thus we can write

$$\delta(du/dx) = d(\delta u)/dx \quad \text{---(i)}$$

$$\delta\left(\int F dx\right) = \int (\delta F) dx \quad \text{---(ii)}$$

$$\delta F = \frac{\partial F}{\partial u} \delta u + \frac{\partial F}{\partial u'} \delta u' + \frac{\partial F}{\partial u''} \delta u'' \quad \text{---(iii)}$$

We know from differential calculus that the necessary condition for a function to be maximum or minimum is that its first derivative must be zero. Similarly, a functional I to be stationary (maximum or minimum), its first variation must be zero.

i.e. $\delta I = 0$

Taking the first variation for the Eqn. (a) and equating to zero yields

$$\delta I = \int_{x_1}^{x_2} \delta F(x, u, u', u'') dx = 0$$

Substitute from Eqn. (iii) we get

$$\delta I = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial u} \delta u + \frac{\partial F}{\partial u'} \delta u' + \frac{\partial F}{\partial u''} \delta u'' \right) dx = 0$$

$$\delta I = \int_{x_1}^{x_2} \frac{\partial F}{\partial u} \delta u dx + \int_{x_1}^{x_2} \frac{\partial F}{\partial u'} \frac{d(\delta u)}{dx} dx + \int_{x_1}^{x_2} \frac{\partial F}{\partial u''} \frac{d^2(\delta u)}{dx^2} dx = 0$$

$$\left| \begin{aligned} \delta u' &= \delta \left(\frac{du}{dx} \right) = \frac{d(\delta u)}{dx} \\ \delta u'' &= \frac{d^2(\delta u)}{dx^2} \end{aligned} \right.$$

---(b)

In order to eliminate $\delta u'$ and $\delta u''$ from Eqn. (b), the second and third terms are integrated by parts.

$$\text{Term-2: } \int_{x_1}^{x_2} \frac{\partial F}{\partial u'} \frac{d(\delta u)}{dx} dx = \left[\frac{\partial F}{\partial u'} \delta u \right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx} \left(\frac{\partial F}{\partial u'} \right) \delta u dx$$

$$\text{Term-3: } \int_{x_1}^{x_2} \frac{\partial F}{\partial u''} \frac{d^2(\delta u)}{dx^2} dx = \left[\frac{\partial F}{\partial u''} \frac{d(\delta u)}{dx} \right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx} \left(\frac{\partial F}{\partial u''} \right) \frac{d(\delta u)}{dx} dx$$

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial u''} \frac{d^2(\delta u)}{dx^2} dx = \left[\frac{\partial F}{\partial u''} \frac{d(\delta u)}{dx} \right]_{x_1}^{x_2} - \left[\frac{d}{dx} \left(\frac{\partial F}{\partial u''} \right) \delta u \right]_{x_1}^{x_2} + \int_{x_1}^{x_2} \frac{d^2}{dx^2} \left(\frac{\partial F}{\partial u''} \right) \delta u dx$$

Substitute into Eqn. (b) and rearranging we get

$$\delta I = \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial u} - \frac{d}{dx} \left(\frac{\partial F}{\partial u'} \right) + \frac{d^2}{dx^2} \left(\frac{\partial F}{\partial u''} \right) \right] \delta u dx + \left[\frac{\partial F}{\partial u'} - \frac{d}{dx} \left(\frac{\partial F}{\partial u''} \right) \right] \delta u \Big|_{x_1}^{x_2} + \left[\left(\frac{\partial F}{\partial u''} \right) \delta u' \right]_{x_1}^{x_2} = 0$$

Since the variation δu is arbitrary, the terms must vanish individually in order to satisfy the above equation.

$$\left[\frac{\partial F}{\partial u} - \frac{d}{dx} \left(\frac{\partial F}{\partial u'} \right) + \frac{d^2}{dx^2} \left(\frac{\partial F}{\partial u''} \right) \right] = 0 \quad \text{---(1)}$$

$$\left[\frac{\partial F}{\partial u'} - \frac{d}{dx} \left(\frac{\partial F}{\partial u''} \right) \right] \delta u \Big|_{x_1}^{x_2} = 0 \quad \text{---(2a)}$$

$$\left[\left(\frac{\partial F}{\partial u''} \right) \delta u' \right]_{x_1}^{x_2} = 0 \quad \text{---(2b)}$$

Eqn. (1) is the governing differential equation for the problem and is called *Euler-Lagrange* equation.

Eqn. 2(a) and 2(b) give the boundary conditions. The conditions

$$\left[\frac{\partial F}{\partial u'} - \frac{d}{dx} \left(\frac{\partial F}{\partial u''} \right) \right]_{x_1}^{x_2} = 0 \quad \text{---(3)}$$

$$\left[\left(\frac{\partial F}{\partial u''} \right) \right]_{x_1}^{x_2} = 0$$

Eqn. (3) are called *natural boundary conditions*.

If one of the natural boundary conditions is not satisfied, the corresponding portions of Eqns. 2(a) and 2(b) must be satisfied.

$$\text{i.e. } \delta u(x_1) = 0, \delta u(x_2) = 0 \quad \text{---(4)}$$

$$\delta u'(x_1) = 0, \delta u'(x_2) = 0$$

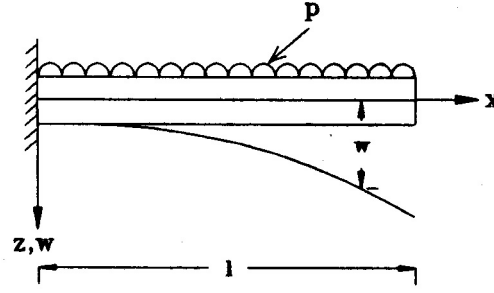
The above conditions (Eqn. (4)) are called *geometric* or *kinematic* or *forced boundary conditions*.

Thus, the calculus of variation provides one governing differential equation (Eqn. (1)) and four boundary conditions (Eqns. (3) and (4)).

Prob. 1: (a) What is functional? Give an example.

(b) Using variational principles, obtain the differential equation and boundary conditions for the beam shown in Fig. The load P is uniformly distributed load.

(University of Mysore, Feb. 2000)



For a beam in bending, the strain energy is given by (see Note)

$$U = \frac{EI}{2} \int_0^l \left(\frac{d^2 w}{dx^2} \right)^2 dx \text{ and work potential due to load } W_P = - \int_0^l P w dx$$

Therefore, total P.E,

$$\pi = \frac{EI}{2} \int_0^l \left(\frac{d^2 w}{dx^2} \right)^2 dx - \int_0^l P w dx = \int_0^l \left[\frac{EI}{2} (w'')^2 - Pw \right] dx \quad \text{---(a)}$$

$$(\because \frac{d^2 w}{dx^2} = w'')$$

Note: The strain energy $U = \frac{1}{2} \iiint \sigma^T \varepsilon dV = \frac{1}{2} \iiint \sigma \varepsilon dx dy dz$ for 1 - D beam $\sigma^T = \sigma$

$$\text{For a beam } \sigma = \frac{M}{I} y = \frac{EI \frac{d^2 w}{dx^2}}{I} y = E \frac{d^2 w}{dx^2} y \quad \because M = EI \frac{d^2 w}{dx^2}$$

$$\text{The strain, } \varepsilon = \sigma/E = \frac{E \frac{d^2 w}{dx^2} y}{E} = \frac{d^2 w}{dx^2} y \quad \therefore \sigma \varepsilon = E \frac{d^2 w}{dx^2} y \frac{d^2 w}{dx^2} y = E \left(\frac{d^2 w}{dx^2} \right)^2 y^2$$

Substitute $\sigma \varepsilon$ into strain energy equation we get

$$U = \frac{1}{2} \int_0^l E \left(\frac{d^2 w}{dx^2} \right)^2 dx \iint y^2 dy dz = \frac{1}{2} \int_0^l E \left(\frac{d^2 w}{dx^2} \right)^2 dx \iint y^2 dA = \int_0^l \frac{EI}{2} \left(\frac{d^2 w}{dx^2} \right)^2 dx$$

where, $I = \iint y^2 dA$ is the second moment of area i.e. moment of inertia I .

Extremizing total P.E (i.e. eqn. (a)) we get

$$\delta \pi = \int_0^l \left(\frac{\partial F}{\partial w} \delta w + \frac{\partial F}{\partial w'} \delta w' + \frac{\partial F}{\partial w''} \delta w'' \right) dx \quad \text{---(b)}$$

$$\text{where, } F = \frac{EI}{2} (w'')^2 - Pw \text{ (from Eqn.a)}$$

Differentiate above equation w.r.t w, w', w''

$$\frac{\partial F}{\partial w} = -P; \frac{\partial F}{\partial w'} = 0 \text{ and } \frac{\partial F}{\partial w''} = EI(w'') \quad \text{---(c)}$$

Substitute Eqn. (c) into Eqn. (b)

$$\delta\pi = \int_0^l [EI(w'')\delta w'' - P\delta w]dx = 0 \quad \text{---(d)}$$

To eliminate $\delta w''$ integrate 1st term of Eqn. (d) by parts

$$\begin{aligned} \int_0^l EI(w'')\delta w'' dx &= EI(w'')\delta w' \Big|_0^l - \int_0^l EI \frac{d}{dx}(w'')\delta w' dx \\ \int_0^l EI(w'')\delta w'' dx &= EI(w'')\delta w' \Big|_0^l - EI \frac{d(w'')}{dx} \delta w \Big|_0^l + \int_0^l EI \frac{d^2(w'')}{dx^2} \delta w dx \end{aligned}$$

Substitute into Eqn. (d) and rearrange we get

$$\delta\pi = EI(w'')\delta w' \Big|_0^l - EI \frac{d(w'')}{dx} \delta w \Big|_0^l + \int_0^l \left[EI \frac{d^2(w'')}{dx^2} - P \right] \delta w dx = 0$$

To satisfy the above condition, the terms must be vanished individually and hence it yields

1. $EI \frac{d^2(w'')}{dx^2} - P = 0$ or $EI w^{iv} = P$ is the differential equation of equilibrium Euler-Lagrange equation.

The other two conditions to be satisfied is at free end of the beam (i.e. at $x = l$) and they are natural boundary conditions

$$2. \quad EI \frac{d(w'')}{dx^2} = EI \frac{d^3 w}{dx^3} = 0 \quad \text{---(e)}$$

$$EI(w'') = EI \frac{d^2 w}{dx^2} = 0 \quad \text{---(f)}$$

We know from strength of material that Eqns. (e) and (f) represents shear force and bending moment and they are zero at free end (i.e. at $x = l$).

The other two boundary conditions to be satisfied at fixed end (at $x = 0$) are

$$w(x=0) = 0 \quad \text{and} \quad w'(x=0) = 0$$

That is the deflection and slop at fixed end of cantilever beam is zero.

Note: The above equations can also be obtained directly using the expressions for Euler – Lagrange equation and natural boundary conditions as follows.

The differential equation (Euler – Lagrange equation) for this case can be formed from

$$\left[\frac{\partial F}{\partial w} - \frac{d}{dx} \left(\frac{\partial F}{\partial w'} \right) + \frac{d^2}{dx^2} \left(\frac{\partial F}{\partial w''} \right) \right] = 0 \quad \text{---(b)}$$

$$\text{where, } F = \frac{EI}{2}(w'')^2 - Pw$$

Differentiate the above equation w.r.t w, w' and w'' we get the Eqn. (c).

Substitute Eqn. (c) into Eqn. (b) we get

$$\left[-P - 0 + EI \frac{d^2(w'')}{dx^2} \right] = 0$$

$EI w^{iv} = P$ is the differential equation of equilibrium of beam.

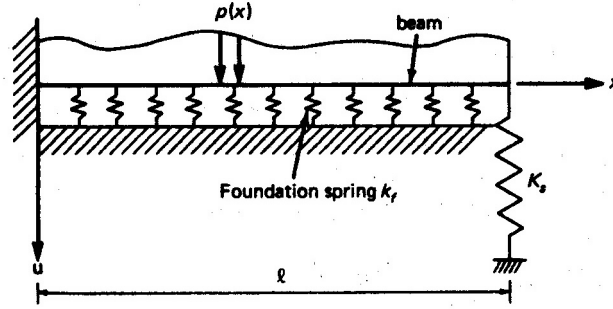
At the free end ($x = l$), the natural boundary must be satisfied and are obtained from

$$\left[\frac{\partial F}{\partial w'} - \frac{d}{dx} \left(\frac{\partial F}{\partial w''} \right) \right] = 0 \quad \text{and} \quad \left(\frac{\partial F}{\partial w''} \right) = 0$$

Substitute from Eqn. (c) into the above equations we get Eqns. (e) and (f).

Prob.2: Determine the total potential for a beam resting on an elastic foundation, with one end rigidly fixed and the other end propped on an elastic support as shown in Fig. assume K_f is the subgrade reaction-spring constant per unit length and K_s is the spring constant for the flexible end support. Using the variational principles, obtain the governing differential equation and boundary conditions for this beam.

(University of Mysore, Feb. 1999)



Solution:

The strain energy of the system is, $U = S.E|_{Beam} + S.E|_{Springs}$

$$S.E|_{Beam} = \frac{EI}{2} \int_0^l \left(\frac{d^2 w}{dx^2} \right)^2 dx = \int_0^l \frac{EI}{2} (w'')^2 dx \quad \because \frac{d^2 w}{dx^2} = w''$$

As K_f is the spring stiffness per unit length and K_s is the spring stiffness at free end, the strain energy of springs is expressed as.

$$S.E|_{Springs} = + \int_0^l \frac{1}{2} K_f w^2 dx + \frac{1}{2} K_s w^2(l) = + \int_0^l \frac{1}{2} K_f w^2 dx + \frac{1}{2} K_s w^2(l)$$

where, $w(l)$ = deflection of beam corresponds to the point of spring support K_s (i.e. at free end). The work potential due to distributed load is expressed as

$$W_P = - \int_0^l P w dx. \text{ Therefore, total P.E is}$$

$$\pi = \int_0^l \left(\frac{EI}{2} (w'')^2 + \frac{1}{2} K_f w^2 - P w \right) dx + \frac{1}{2} K_s w^2(l)$$

Extremizing the above equation, we get

$$\delta \pi = \int_0^l \left(\frac{\partial F}{\partial w} \delta w + \frac{\partial F}{\partial w'} \delta w' + \frac{\partial F}{\partial w''} \delta w'' \right) dx + \frac{\partial G}{\partial w(l)} \delta w(l) = 0 \quad \text{---(a)}$$

$$\text{where, } F = \frac{EI}{2} (w'')^2 + \frac{1}{2} K_f w^2 - P w \text{ and } G = + \frac{1}{2} K_s w^2(l)$$

$$\therefore \frac{\partial F}{\partial w} = K_f w - P; \frac{\partial F}{\partial w'} = 0; \frac{\partial F}{\partial w''} = EI(w'') \text{ and } \frac{\partial G}{\partial w(l)} = K_s w(l)$$

Substitute into Eqn. (a) we get

$$\delta \pi = \int_0^l \left(EI(w'') \delta w'' + (K_f w - P) \delta w \right) dx + K_s w(l) \delta w(l) = 0 \quad \text{---(b)}$$

To eliminate $\delta w''$, integrate 1st term of the above equation by parts

i.e.

$$\int_0^l EI(w'')\delta w'' dx = EI(w'')\delta w'|_0^l - \int_0^l EI \frac{d(w'')}{dx} \delta w' dx = EI(w'')\delta w'|_0^l - EI \frac{d(w'')}{dx} \delta w \Big|_0^l + \int_0^l EI \frac{d^2(w'')}{dx^2} \delta w dx$$

Substitute into Eqn. (b) and rearranging we get

$$\delta \pi = EI(w'')\delta w'|_0^l - EI \frac{d(w'')}{dx} \delta w \Big|_0^l + \int_0^l \left(EI \frac{d^2(w'')}{dx^2} + (K_f w - P) \right) \delta w dx + K_s w(l) \delta w(l) = 0$$

$$\delta \pi = + \int_0^l \left(EI \frac{d^2(w'')}{dx^2} + (K_f w - P) \right) \delta w dx + EI(w'')\delta w'|_0^l - EI \frac{d(w'')}{dx} \delta w \Big|_0^l + K_s w(l) \delta w(l) = 0 \text{ --- (c)}$$

The above equation yields

1. Differential equation of the bar

$$EI \frac{d^2(w'')}{dx^2} + (K_f w - P) = 0 \text{ or } EI \frac{d^4 w}{dx^4} = P - K_f w \quad \text{at } 0 < x < l$$

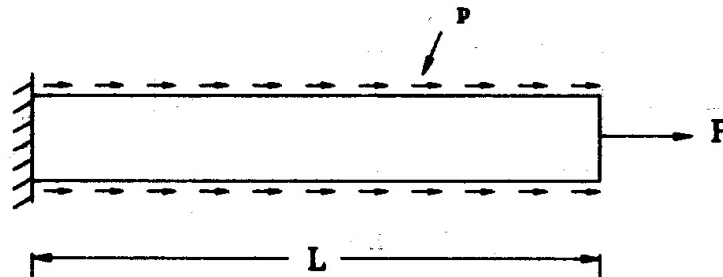
2. The natural boundary conditions at free end (i.e. at $x = l$) are

$$-EI \frac{d(w'')}{dx} + K_s w(l) = -EI \frac{d^3 w}{dx^3} + K_s w(l) = 0 \text{ (sum of coefficient of } \delta w \text{ and } \delta w(l) \text{ in eq. (c))}$$

$$EI w'' = EI \frac{d^2 w}{dx^2} = 0 \text{ (the coefficient of } \delta u').$$

Two more B.C at $x = 0$ are $\delta w = \delta w' = 0$.

Prob.3: An axial bar shown in Fig. is fixed at one end. It is subjected to a distributed axial force p / unit length and a force P at the free end. Calculate the total potential energy of the bar. Using the variational principles, derive the governing differential equation and boundary conditions for the bar.



The strain energy of the bar is given by

$$U = \int_0^l \frac{AE}{2} \left(\frac{du}{dx} \right)^2 dx$$

$$U = \frac{1}{2} \int_V \sigma^T \epsilon dV = \frac{1}{2} \int_0^l \sigma \epsilon A dx$$

$$\text{But } \sigma = E \epsilon = E \frac{du}{dx} \text{ and hence, } \sigma \epsilon = E \left(\frac{du}{dx} \right)^2$$

Substitute into U we get

$$\text{For 1 - D bar } \sigma^T \epsilon = \sigma \epsilon$$

$$\text{and } dV = A dx$$

$$U = \int_0^l \frac{AE}{2} \left(\frac{du}{dx} \right)^2 dx$$

The work potential is given by

$$W_P = - \int_0^l p u dx - P u(l)$$

where, u is the deformation of the body and $u(l)$ is the deformation of the body at the point of application of concentrated load (i.e. at free end).

The total potential energy is

$$\pi = \int_0^l \frac{AE}{2} \left(\frac{du}{dx} \right)^2 dx - \int_0^l p u dx - P u(l) = \int_0^l \left(\frac{AE}{2} (u')^2 - p u \right) dx - P u(l)$$

Extremizing the above potential energy functional we get

$$\delta\pi = \int_0^l \left(\frac{\partial F}{\partial u} \delta u + \frac{\partial F}{\partial u'} \delta u' + \frac{\partial F}{\partial u''} \delta u'' \right) dx + \frac{\partial G}{\partial u(l)} \delta u(l) = 0 \quad \text{---(a)}$$

where, $F = \frac{AE}{2} (u')^2 - p u$ and $G = -P u(l)$

$$\frac{\partial F}{\partial u} = -p; \quad \frac{\partial F}{\partial u'} = AE(u'); \quad \frac{\partial F}{\partial u''} = 0 \quad \text{and} \quad \frac{\partial G}{\partial u(l)} = -P$$

Substitute into eqn. (a) and rearranging we get

$$\delta\pi = \int_0^l (AE(u') \delta u' - p \delta u) dx - P \delta u(l) = 0 \quad \text{---(b)}$$

To eliminate $\delta u'$ integrate 1st term of the above equation by parts

$$\int_0^l AE(u') \delta u' dx = AE(u') \delta u \Big|_0^l - \int_0^l AE \frac{d(u')}{dx} \delta u dx$$

Substitute into Eqn. (b) we get

$$\delta\pi = AE(u') \delta u \Big|_0^l - \int_0^l \left(AE \frac{d(u')}{dx} \right) \delta u dx - \int_0^l p \delta u dx - P \delta u(l) = 0$$

$$\delta\pi = AE(u') \delta u \Big|_0^l - P \delta u(l) - \int_0^l (AE u'' + p) \delta u dx = 0 \quad \therefore \frac{du'}{dx} = \frac{d}{dx} \left(\frac{du}{dx} \right) = u''$$

The above equation yields

1. The differential equation of the bar and is given by

$$\therefore AE u'' + p = 0 \quad \text{or} \quad AE \frac{d^2 u}{dx^2} = -p \quad \text{at } 0 < x < l$$

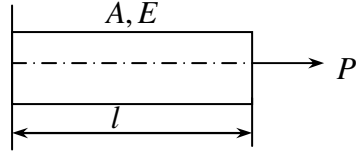
2. Natural boundary condition at free end is given by

$$AE(u') - P = 0 \quad \text{or} \quad AE \frac{du}{dx} = P \quad \text{at } x = l$$

The other boundary condition is $\delta u(x=0) = 0$

Prob.4: An elastic bar of length L , Modulus of elasticity E , area of cross section A , which is fixed at one end and is subjected to axial load at the other end. Obtain the Euler equation governing the bar and natural boundary conditions. 10 marks (VTU, July 2004)

Solution:



The strain energy of the bar is given by

$$U = \int_0^l \frac{AE}{2} \left(\frac{du}{dx} \right)^2 dx$$

The work potential is given by

$$W_P = -Pu(l)$$

where, $u(l)$ is the deformation of the body at the point of application of concentrated load (i.e. at free end).

The total potential energy is

$$\pi = \int_0^l \frac{AE}{2} \left(\frac{du}{dx} \right)^2 dx - Pu(l) = \int_0^l \left(\frac{AE}{2} (u')^2 \right) dx - Pu(l)$$

Extremizing the above potential energy functional we get

$$\delta\pi = \int_0^l \left(\frac{\partial F}{\partial u} \delta u + \frac{\partial F}{\partial u'} \delta u' + \frac{\partial F}{\partial u''} \delta u'' \right) dx + \frac{\partial G}{\partial u(l)} \delta u(l) = 0 \quad \text{---(a)}$$

where, $F = \frac{AE}{2} (u')^2$ and $G = -Pu(l)$

$$\frac{\partial F}{\partial u} = 0; \frac{dF}{du'} = AE(u'); \frac{dF}{du''} = 0 \text{ and } \frac{dG}{du(l)} = -P$$

Substitute into eqn. (a) and rearranging we get

$$\delta\pi = \int_0^l AE(u') \delta u' dx - P \delta u(l) = 0 \quad \text{---(b)}$$

To eliminate $\delta u'$ integrate 1st term of the above equation by parts

$$\int_0^l AE(u') \delta u' dx = AE(u') \delta u \Big|_0^l - \int_0^l AE \frac{d(u')}{dx} \delta u dx$$

Substitute into Eqn. (b) we get

$$\delta\pi = AE(u') \delta u \Big|_0^l - \int_0^l \left(AE \frac{d(u')}{dx} \right) \delta u dx - P \delta u(l) = 0$$

The above equation yields

1. The differential equation of the bar and is given by

$$\therefore -AE \frac{d(u')}{dx} = 0 \text{ or } AE \frac{d^2 u}{dx^2} = 0 \quad \text{at } 0 < x < l$$

2. Natural boundary condition at free end is given by

$$AE(u') - P = 0 \text{ or } AE \frac{du}{dx} = P \quad \text{at } x = l$$

The other boundary condition is $\delta u(x=0) = 0$